Extended and Reshetikhin Twists for $sl(3)$[1]

Vladimir D. Lyakhovsky[2] and Mariano A. del Olmo

Departamento de Física Teórica, Universidad de Valladolid,
E-47011, Valladolid, Spain

email: vladimir@klander.fam.cie.uva.es, olmo@fta.uva.es

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Abstract

The properties of the set $\mathcal{L}$ of extended jordanian twists for algebra $sl(3)$ are studied. Starting from the simplest algebraic construction — the peripheric Hopf algebra $U_{\mathcal{P}'(0,1)}(sl(3))$ — we construct explicitly the complete family of extended twisted algebras $\{U_{\mathcal{E}(\theta)}(sl(3))\}$ corresponding to the set of 4-dimensional Frobenius subalgebras $\{\mathbf{L}(\theta)\}$ in $sl(3)$. It is proved that the extended twisted algebras with different values of the parameter θ are connected by a special kind of Reshetikhin twist. We study the relations between the family $\{U_{\mathcal{E}(\theta)}(sl(3))\}$ and the one-dimensional set $\{U_{\mathcal{DJR}(\lambda)}(sl(3))\}$ produced by the standard Reshetikhin twist from the Drinfeld–Jimbo quantization $U_{\mathcal{DJ}}(sl(3))$. These sets of deformations are in one-to-one correspondence: each element of $\{U_{\mathcal{E}(\theta)}(sl(3))\}$ can be obtained by a limiting procedure from the unique point in the set $\{U_{\mathcal{DJR}(\lambda)}(sl(3))\}$.

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[2] In absence from Theoretical Department, Sankt-Petersburg State University; 198904, St. Petersburg, Russia.

1 Introduction

The triangular Hopf algebras and twists (they preserve the triangularity [1, 2]) play an important role in quantum group theory and applications [3, 4, 5]. Very few types of twists were written explicitly in a closed form. The well known example is the jordanian twist ($\mathcal{JT}$) of the Borel algebra $B(2)$ ($\{H, E | [H, E] = E\}$) with $r = H \wedge E$ [6] where the triangular R–matrix $\mathcal{R} = (\mathcal{F}_{j})_{21}\mathcal{F}_{j}^{\ -1}$ is defined by the twisting element [7, 8]

$$\mathcal{F}_{j} = \exp\{H \otimes \sigma\}, \tag{1.1}$$

with $\sigma = \ln(1 + E)$. In [9] it was shown that there exist different extensions ($\mathcal{ET}$'s) of this twist. Using the notion of factorizable twist [10] the element $\mathcal{F}_{\mathcal{E}} \in \mathcal{U}(sl(N))^{\otimes 2}$,

$$\mathcal{F}_{\mathcal{E}} = \Phi_e \Phi_j = \exp\{2\xi \sum_{i=2}^{N-1} E_{1i} \otimes E_{iN} e^{-\widetilde{\sigma}}\} \exp\{H \otimes \widetilde{\sigma}\}, \tag{1.2}$$

was proved to satisfy the twist equation, where $E = E_{1N}$, $H = E_{11} - E_{NN}$ is one of the Cartan generators $H \in \mathfrak{h}(sl(N))$, $\widetilde{\sigma} = \frac{1}{2}\ln(1 + 2\xi E)$ and $\{E_{ij}\}_{i,j=1,\ldots N}$ is the standard $gl(N)$ basis.

Studying the family $\{\mathbf{L}(\alpha, \beta, \gamma, \delta)_{\alpha+\beta=\delta}\}$ of carrier algebras for extended jordanian twists $\mathcal{F}_{\mathcal{E}(\alpha,\beta,\gamma,\delta)}$ [11] it is sufficient to consider the one-dimensional set $\mathcal{L} = \{\mathbf{L}(\alpha, \beta)_{\alpha+\beta=1,}\}$ (for different nonzero γ's and δ's the Hopf algebras $\mathbf{L}_{\mathcal{E}}$, obtained by the corresponding twistings, are equivalent).

The connection of the Drinfeld–Jimbo ($\mathcal{DJ}$) deformation of a simple Lie algebra $\mathfrak{g}$ [6, 12] with the jordanian deformation was already pointed out in [8]. The similarity transformation of the classical r–matrix

$$r_{\mathcal{DJ}} = \sum_{i=1}^{\mathrm{rank}(\mathfrak{g})} t_{ij} H_i \otimes H_j + \sum_{\alpha \in \Delta_+} E_\alpha \otimes E_{-\alpha}$$

performed by the operator $\exp(v \ \mathrm{ad}_{E_{1N}})$ turns $r_{\mathcal{DJ}}$ into the sum $r_{\mathcal{DJ}} + v \ r_j$ [8] where

$$r_j = -v \left(H_{1N} \wedge E_{1N} + 2 \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN} \right). \tag{1.3}$$

Hence, r_j is also a classical r–matrix and defines the corresponding deformation. A contraction of the quantum Manin plane $xy = qyx$ of $\mathcal{U}_{\mathcal{DJ}}(sl(2))$ with the mentioned above similarity transformation in the fundamental representation $M = 1 + v(1-q)^{-1}\rho(E_{12})$ results in the jordanian plane $x'y' = y'x' + vy'^2$ of $\mathcal{U}_j(sl(2))$ [7]. Thus, the canonical extended jordanian twisted algebra $U_{\mathcal{E}(1/2,1/2)}$, which corresponds in our notation to the carrier subalgebra $\mathbf{L}_{(1/2,1/2)}$, can be treated as a limit case for the parameterized set of Drinfeld–Jimbo quantizations. Contrary to this fact other extended twists of $U(sl(N))$ do not reveal such properties with respect to the standard deformation. In particular,

the $U_{\mathcal{P}}(sl(4))$ algebra twisted by the so-called peripheric twist ($\mathcal{PT}$) was found to be disconnected with the Drinfeld–Jimbo deformation $U_{\mathcal{DJ}}(sl(4))$.

In this paper we study the properties of the deformations induced in $U(sl(3))$ by the set of extended twists $\mathcal{F}_{\mathcal{E}(\alpha,\beta)}$. We consider the deformations of simple Lie algebras. So, the parameters α and β (arising from the reparametrization of the root space) can be treated as belonging to $\mathbf{R}^1$. The same is true for other parameters $(\lambda, \theta, \ldots)$ appearing in this study. In the twist equivalence transformations they can be considered as belonging to $\mathbf{C}^1$. But in the present approach it is sufficient to treat them as real numbers.

We show that to any Hopf algebra $U_{\mathcal{E}(\alpha,\beta)}$ one can apply additional Reshetikhin twist [13] $\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)}$ whose (abelian) carrier subalgebra is generated by $K \in \mathfrak{h}(sl(N))$ and $E \in \mathbf{L}$:

$$U_{\mathcal{E}(\alpha,\beta)} \stackrel{\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)}}{\longrightarrow} U_{\mathcal{E}\widetilde{\mathcal{R}}(\alpha,\beta,\lambda)}. \tag{1.4}$$

However, the carrier subalgebra of $\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)} \circ \mathcal{F}_{\mathcal{E}(\alpha,\beta)}$ is the same as for $\mathcal{F}_{\mathcal{E}(\alpha,\beta)}$ because of the isomorphism:

$$U_{\mathcal{E}\widetilde{\mathcal{R}}(\alpha,\beta,\lambda)} \approx U_{\mathcal{E}(\alpha+\lambda,\beta-\lambda)}. \tag{1.5}$$

Twists $\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)}$ act transitively on the set $\{U_{\mathcal{E}(\alpha,\beta)}\}$. Simultaneously we consider the canonical Reshetikhin twist $\mathcal{F}_{\mathcal{R}(\theta)} = e^{\theta H_1 \otimes H_2}$ [13] that performs the transition from $U_{\mathcal{DJ}}(sl(3))$ to the parametric quantization:

$$U_{\mathcal{DJ}} \stackrel{\mathcal{F}_{\mathcal{R}(\theta)}}{\longrightarrow} U_{\mathcal{DJR}(\theta)}. \tag{1.6}$$

It is worth mentioning that in the case of $U_{\mathcal{DJ}}(gl(3))$ such kind of transformations can be used to obtain possibilities for additional twistings [14].

Finally, the two sets of parameterized Lie algebras are formed: $\{\mathfrak{g}^*_{\mathcal{E}}\ (\lambda)\}$ and $\{\mathfrak{g}^*_{\mathcal{DJR}}\ (\theta)\}$. The elements of both of them are dual to $sl(3)$. Using the technique elaborated in [11, 15] we prove a one-to-one correspondence between the members of these sets: for any λ_0 fixed there is one and only one θ_0 such that $\mathfrak{g}^*_{\mathcal{E}}\ (\lambda_0)$ and $\mathfrak{g}^*_{\mathcal{DJR}}\ (\theta_0)$ are the first order deformations of each other. This means that for any $U_{\mathcal{E}(\alpha,\beta)}(sl(3))$ there exists one and only one such $U_{\mathcal{DJR}(\theta)}(sl(3))$ that these two Hopf algebras can be connected by a smooth sequence of quantized Lie bialgebras.

In Section 2 we present a short list of basic relations for twists. The general properties of extended twists for $U(sl(3))$ are displayed in Section 3. There we construct explicitly the peripheric extended twisted algebra $U_{\mathcal{P}'}(sl(3))$. In Section 4 the special kind of Reshetikhin twist for $U_{\mathcal{P}'}(sl(3))$ is composed and as a result the family $\{U_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(sl(3))\}$ is obtained. We prove that this solves the problem of finding the whole set $\{U_{\mathcal{E}}(sl(3))\}$ of extended twists. The relations between the multiparametric $\mathcal{DJ}$ quantizations and twisted algebras $\{U_{\mathcal{E}}(sl(3))\}$ are studied in Section 5, and their one-to-one correspondence is established. The defining relations for the canonically extended twisted algebra $U^{\mathrm{can}}_{\mathcal{E}}(sl(3))$ are presented in the Appendix.

2 Basic definitions

In this section we remind briefly the basic properties of twists.

A Hopf algebra $\mathcal{A}(m, \Delta, \epsilon, S)$ with multiplication $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, coproduct $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\epsilon\colon \mathcal{A} \to C$, and antipode $S : \mathcal{A} \to \mathcal{A}$ can be transformed [1] by an invertible (twisting) element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, $\mathcal{F} = \sum f_i^{(1)} \otimes f_i^{(2)}$, into a twisted one $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}})$. This Hopf algebra $\mathcal{A}_{\mathcal{F}}$ has the same multiplication and counit but the twisted coproduct and antipode given by

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \qquad S_{\mathcal{F}}(a) = V S(a) V^{-1}, \tag{2.1}$$

with

$$V = \sum f_i^{(1)} S(f_i^{(2)}), \qquad a \in \mathcal{A}.$$

The twisting element has to satisfy the equations

$$(\epsilon \otimes id)(\mathcal{F}) = (id \otimes \epsilon)(\mathcal{F}) = 1, \tag{2.2}$$

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}). \tag{2.3}$$

The first one is just a normalization condition and follows from the second relation modulo a non-zero scalar factor.

If $\mathcal{A}$ is a Hopf subalgebra of $\mathcal{B}$ the twisting element $\mathcal{F}$ satisfying (2.1)–(2.3) induces the twist deformation $\mathcal{B}_{\mathcal{F}}$ of $\mathcal{B}$. In this case one can put $a \in \mathcal{B}$ in all the formulas (2.1). This will completely define the Hopf algebra $\mathcal{B}_{\mathcal{F}}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the universal enveloping algebras: $\mathcal{A} = U(\mathfrak{l}) \subset \mathcal{B} = U(\mathfrak{g})$ with $\mathfrak{l} \subset \mathfrak{g}$. If $U(\mathfrak{l})$ is the minimal subalgebra on which $\mathcal{F}$ is completely defined as $\mathcal{F} \in U(\mathfrak{l}) \otimes U(\mathfrak{l})$ then $\mathfrak{l}$ is called the carrier algebra for $\mathcal{F}$ [8].

The composition of appropriate twists can be defined as $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1$. Here the element $\mathcal{F}_1$ has to satisfy the twist equation with the coproduct of the original Hopf algebra, while $\mathcal{F}_2$ must be its solution for $\Delta_{\mathcal{F}_1}$ of the algebra twisted by $\mathcal{F}_1$.

If the initial Hopf algebra $\mathcal{A}$ is quasitriangular with the universal element $\mathcal{R}$ then so is the twisted one $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ with

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\, \mathcal{R}\, \mathcal{F}^{-1}. \tag{2.4}$$

Most of the explicitly known twisting elements have the factorization property with respect to comultiplication

$$(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}\mathcal{F}_{13} \qquad \text{or} \qquad (\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}\,,$$

and

$$(id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13} \qquad \text{or} \qquad (id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12}\,.$$

To guarantee the validity of the twist equation, these identities are to be combined with the additional requirement $\mathcal{F}_{12}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{12}$ or the Yang–Baxter equation on $\mathcal{F}$ [10].

An important subclass of factorizable twists consists of elements satisfying the equations

$$(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}\,, \tag{2.5}$$

$$(id \otimes \Delta_{\mathcal{F}})(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}\,. \tag{2.6}$$

Apart from the universal R–matrix $\mathcal{R}$ that satisfies these equations for $\Delta_{\mathcal{F}} = \Delta^{op}$ ($\Delta^{op} = \tau \circ \Delta$, where $\tau(a \otimes b) = b \otimes a$) there are two more well developed cases of such twists: the jordanian twist of a Borel algebra $B(2)$ where $\mathcal{F}_j$ has the form (1.1) (see [7]) and the extended jordanian twists (see [9] and [11] for details).

According to the result by Drinfeld [2] skew (constant) solutions of the classical Yang–Baxter equation (CYBE) can be quantized and the deformed algebras thus obtained can be presented in a form of twisted universal enveloping algebras. On the other hand, such solutions of CYBE can be connected with the quasi-Frobenius carrier subalgebras of the initial classical Lie algebra [16]. A Lie algebra $\mathfrak{g}(\mu)$, with the Lie composition μ, is called Frobenius if there exists a linear functional $g^* \in \mathfrak{g}^*$ such that the form $b(g_1, g_2) = g^*(\mu(g_1, g_2))$ is nondegenerate. This means that $\mathfrak{g}$ must have a nondegenerate 2–coboundary $b(g_1, g_2) \in B^2(\mathfrak{g}, \mathbf{K})$. The algebra is called quasi-Frobenius if it has a nondegenerate 2–cocycle $b(g_1, g_2) \in Z^2(\mathfrak{g}, \mathbf{K})$ (not necessarily a coboundary). The classification of quasi-Frobenius subalgebras in $sl(n)$ was given in [16].

The deformations of quantized algebras include the deformations of their Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$. Let us fix one of the constituents $\mathfrak{g}_1^*(\mu_1^*)$ (with composition μ_1^*) and deform it in the first order

$$(\mu_1^*)_t = \mu_1^* + t\mu_2^*,$$

its deforming function μ_2^* is also a Lie product and the deformation property becomes reciprocal: μ_1^* can be considered as a first order deforming function for algebra $\mathfrak{g}_2^*(\mu_2^*)$. Let $\mathfrak{g}(\mu)$ be a Lie algebra that form Lie bialgebras with both $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$. This means that we have a one-dimensional family $\{(\mathfrak{g}, (\mathfrak{g}_1^*)_t)\}$ of Lie bialgebras and correspondingly a one dimensional family of quantum deformations $\{\mathcal{A}_t(\mathfrak{g}, (\mathfrak{g}_1^*)_t)\}$ [17]. This situation provides the possibility to construct in the set of Hopf algebras a smooth curve connecting quantizations of the type $\mathcal{A}(\mathfrak{g}, \mathfrak{g}_1^*)$ with those of $\mathcal{A}(\mathfrak{g}, \mathfrak{g}_2^*)$. Such smooth transitions can involve contractions provided $\mu_2^* \in B^2(\mathfrak{g}_1^*, \mathfrak{g}_1^*)$. This happens in the case of $\mathcal{JT}$, $\mathcal{ET}$ and some other twists (see [15] and references therein).

3 Extended twist for $U(sl(3))$

Extended jordanian twists are associated with the set $\{\mathbf{L}(\alpha, \beta, \gamma, \delta)_{\alpha+\beta=\delta}\}$ of Frobenius algebras [9],[11]

$$\begin{array}{lll} [H, E] = \delta E, & [H, A] = \alpha A, & [H, B] = \beta B, \\ [A, B] = \gamma E, & [E, A] = [E, B] = 0, & \quad \alpha + \beta = \delta. \end{array} \tag{3.1}$$

For limit values of γ and δ the structure of $\mathbf{L}$ degenerates. For the internal (nonzero) values of γ and δ the twists associated with the corresponding $\mathbf{L}$'s are equivalent. It is sufficient to study the one-dimensional subvariety $\mathcal{L} = \{\mathbf{L}(\alpha,\beta)_{\alpha+\beta=1}\}$, that is to consider the carrier algebras

$$\begin{array}{l} [H,E]=E, \quad [H,A]=\alpha A, \quad [H,B]=\beta B, \\ [A,B]=E, \quad [E,A]=[E,B]=0, \qquad \alpha+\beta=1. \end{array} \tag{3.2}$$

The corresponding group 2–cocycles (twists) are

$$\mathcal{F}_{\mathcal{E}(\alpha,\beta)} = \Phi_{\mathcal{E}(\alpha,\beta)}\Phi_j \tag{3.3}$$

or

$$\mathcal{F}_{\mathcal{E}'(\alpha,\beta)} = \Phi_{\mathcal{E}'(\alpha,\beta)}\Phi_j \tag{3.4}$$

with

$$\begin{array}{lcl} \Phi_j & = & \mathcal{F}_j = \exp\{H\otimes\sigma\}, \\ \Phi_{\mathcal{E}(\alpha,\beta)} & = & \exp\{A\otimes Be^{-\beta\sigma}\}, \\ \Phi_{\mathcal{E}'(\alpha,\beta)} & = & \exp\{-B\otimes Ae^{-\alpha\sigma}\}. \end{array} \tag{3.5}$$

Twists (3.3) and (3.4) define the deformed Hopf algebras $\mathbf{L}_{\mathcal{E}(\alpha,\beta)}$ with the co-structure

$$\begin{array}{lcl} \Delta_{\mathcal{E}(\alpha,\beta)}(H) & = & H\otimes e^{-\sigma}+1\otimes H - A\otimes Be^{-(\beta+1)\sigma}, \\ \Delta_{\mathcal{E}(\alpha,\beta)}(A) & = & A\otimes e^{-\beta\sigma}+1\otimes A, \\ \Delta_{\mathcal{E}(\alpha,\beta)}(B) & = & B\otimes e^{\beta\sigma}+e^{\sigma}\otimes B, \\ \Delta_{\mathcal{E}(\alpha,\beta)}(E) & = & E\otimes e^{\sigma}+1\otimes E; \end{array} \tag{3.6}$$

and $\mathbf{L}_{\mathcal{E}'(\alpha,\beta)}$ defined by

$$\begin{array}{lcl} \Delta_{\mathcal{E}'(\alpha,\beta)}(H) & = & H\otimes e^{-\sigma}+1\otimes H + B\otimes Ae^{-(\alpha+1)\sigma}, \\ \Delta_{\mathcal{E}'(\alpha,\beta)}(A) & = & A\otimes e^{\alpha\sigma}+e^{\sigma}\otimes A, \\ \Delta_{\mathcal{E}'(\alpha,\beta)}(B) & = & B\otimes e^{-\alpha\sigma}+1\otimes B, \\ \Delta_{\mathcal{E}'(\alpha,\beta)}(E) & = & E\otimes e^{\sigma}+1\otimes E. \end{array} \tag{3.7}$$

The sets $\{\mathbf{L}_{\mathcal{E}(\alpha,\beta)}\}$ and $\{\mathbf{L}_{\mathcal{E}'(\alpha,\beta)}\}$ are equivalent due to the Hopf isomorphism $\mathbf{L}_{\mathcal{E}(\alpha,\beta)} \approx \mathbf{L}_{\mathcal{E}'(\beta,\alpha)}$:

$$\{\mathbf{L}_{\mathcal{E}}(\alpha,\beta)\} \approx \{\mathbf{L}_{\mathcal{E}'}(\alpha,\beta)\} \approx \{\mathbf{L}_{\mathcal{E}}(\alpha\geq\beta)\} \cup \{\mathbf{L}_{\mathcal{E}'}(\alpha\geq\beta)\}. \tag{3.8}$$

So, it is sufficient to use only one of the extensions either $\Phi_{\mathcal{E}(\alpha,\beta)}$ or $\Phi_{\mathcal{E}'(\alpha,\beta)}$, or a half of the domain for (α,β).

The set $\mathcal{L} = \{\mathbf{L}(\alpha,\beta)_{\alpha+\beta=1,}\}$ is just the family of 4-dimensional Frobenius algebras that one finds in $U(sl(3))$ [16]. It was mentioned in [9] that complicated calculations are needed to write down all the defining coproducts for the canonical extended twisted

$U_{\mathcal{E}}^{\rm can}(sl(3))$. Here we shall show how to overcome partially this difficulty and to get all the defining relations in the explicit form.

First we shall construct the simplest member of the family $\{U_{\mathcal{E}(\alpha,\beta)}(sl(3))\}$ — one of the peripheric twisted algebras $U_{\mathcal{P}'}(sl(3))$. Then, the additional parameterized twist will be applied and finally we shall prove that the whole set $\{U_{\mathcal{E}(\alpha,\beta)}(sl(3))\}$ is thus obtained.

Consider the subalgebra $\mathbf{L}(0,1) \subset sl(3)$ with generators

$$\begin{array}{l} H = \frac{1}{3}(H_{13} + H_{23}) = \frac{1}{3}(E_{11} + E_{22} - 2E_{33}), \\ A = E_{12}, \quad B = E_{23}, \quad E = E_{13}, \end{array} \tag{3.9}$$

and the compositions

$$\begin{array}{l} [H, E_{13}] = E_{13}, \quad [H, E_{12}] = 0, \quad [H, E_{23}] = E_{23}, \\ {[E_{12}, E_{23}]} = E_{13}, \quad [E_{12}, E_{13}] = [E_{23}, E_{13}] = 0. \end{array} \tag{3.10}$$

According to the results obtained in [11] (see formulas (3.4) and (3.5)) one of the peripheric twists attributed to this algebra has the form

$$\mathcal{F}_{\mathcal{P}'} = \Phi_{\mathcal{P}'}\Phi_j = e^{-E_{23}\otimes E_{12}} e^{H\otimes\sigma}. \tag{3.11}$$

Applying to $U(sl(3))$ the twisting procedure with $\mathcal{F}_{\mathcal{P}'}$ we construct the Hopf algebra $U_{\mathcal{P}'}(sl(3))$ with the usual multiplication of $U(sl(3))$ and the coproduct defined by the relations:

$$\begin{array}{rcl} \Delta_{\mathcal{P}'}(H_{12}) &=& H_{12}\otimes 1 + 1\otimes H_{12} + H\otimes(e^{-\sigma}-1) + E_{23}\otimes E_{12}e^{-\sigma}, \\ \Delta_{\mathcal{P}'}(H_{13}) &=& H_{13}\otimes 1 + 1\otimes H_{13} + 2H\otimes(e^{-\sigma}-1) + 2E_{23}\otimes E_{12}e^{-\sigma}, \\ \Delta_{\mathcal{P}'}(E_{12}) &=& E_{12}\otimes 1 + e^{\sigma}\otimes E_{12}, \\ \Delta_{\mathcal{P}'}(E_{13}) &=& E_{13}\otimes e^{\sigma} + 1\otimes E_{13}, \\ \Delta_{\mathcal{P}'}(E_{21}) &=& E_{21}\otimes 1 + 1\otimes E_{21} - H\otimes E_{23}e^{-\sigma} - E_{23}\otimes H_{12} \\ && -\, E_{23}\otimes E_{12}E_{23}e^{-\sigma} + HE_{23}\otimes(1-e^{-\sigma}) - E_{23}^2\otimes E_{12}e^{-\sigma}, \\ \Delta_{\mathcal{P}'}(E_{23}) &=& E_{23}\otimes 1 + 1\otimes E_{23}, \\ \Delta_{\mathcal{P}'}(E_{31}) &=& E_{31}\otimes e^{-\sigma} + 1\otimes E_{31} + H\otimes H_{13} \\ && +\,(1-H)H\otimes(e^{-\sigma}-e^{-2\sigma}) \\ && +\,(1-H)E_{23}\otimes E_{12}(e^{-\sigma}-2e^{-2\sigma}) - E_{21}\otimes E_{12}e^{-\sigma} \\ && +\,E_{23}\otimes E_{32} + E_{23}\otimes H_{13}E_{12}e^{-\sigma} + E_{23}^2\otimes E_{12}^2e^{-2\sigma}, \\ \Delta_{\mathcal{P}'}(E_{32}) &=& E_{32}\otimes e^{-\sigma} + 1\otimes E_{32} + (H-H_{23})\otimes E_{12}e^{-\sigma}. \end{array} \tag{3.12}$$

The universal $\mathcal{R}$–matrix for this algebra is

$$\mathcal{R}_{\mathcal{P}'} = e^{-E_{12}\otimes E_{23}} e^{\sigma\otimes H} e^{-H\otimes\sigma} e^{E_{23}\otimes E_{12}}, \tag{3.13}$$

and the classical r–matrix can be written in the form

$$r_{\mathcal{P}'} = E_{23} \wedge E_{12} + \frac{1}{3} E_{13} \wedge (E_{11} + E_{22} - 2E_{33}). \tag{3.14}$$

By means of this r–matrix (or directly from the coproducts (3.12)) the following Lie compositions for $\mathfrak{g}^*_{\mathcal{P}'}$ (the algebra dual to $sl(3)$ in this quantization) can be obtained

$$\begin{array}{lcllcl}
[X_{11}, X_{13}] & = & -\frac{1}{3}(X_{11} - X_{33}), & [X_{12}, X_{23}] & = & -(X_{11} - X_{33}), \\
[X_{22}, X_{13}] & = & -\frac{1}{3}(X_{11} - X_{33}), & [X_{12}, X_{13}] & = & -X_{12}, \\
[X_{33}, X_{13}] & = & +\frac{2}{3}(X_{11} - X_{33}), & [X_{12}, X_{21}] & = & X_{31}, \\
[X_{11}, X_{23}] & = & +\frac{2}{3} X_{21}, & [X_{13}, X_{31}] & = & X_{31}, \\
[X_{22}, X_{23}] & = & -\frac{4}{3} X_{21}, & [X_{23}, X_{32}] & = & X_{31}, \\
[X_{33}, X_{23}] & = & +\frac{2}{3} X_{21}, & [X_{13}, X_{32}] & = & +X_{32}, \\
[X_{11}, X_{33}] & = & +\frac{1}{3} X_{31}, & [X_{22}, X_{33}] & = & -\frac{1}{3} X_{31}, \\
[X_{11}, X_{12}] & = & +\frac{1}{3} X_{32}, & [X_{12}, X_{33}] & = & -\frac{1}{3} X_{32}, \\
[X_{11}, X_{22}] & = & -\frac{1}{3} X_{31}, & [X_{12}, X_{22}] & = & +\frac{2}{3} X_{32}.
\end{array} \tag{3.15}$$

4 Reshetikhin twist action on $U_{\mathcal{E}}(sl(3))$

The main observation with respect to our present aim is that besides the primitive element σ the twisted algebra $U_{\mathcal{P}'}(sl(3))$ contains the primitive Cartan generator K

$$K = \frac{1}{3}(H_{12} - H_{23}). \tag{4.1}$$

The element K^* dual to K is orthogonal to the root E^* of $E \in \mathbf{L}(\alpha, \beta)$, that is, K commutes with σ. So $U_{\mathcal{P}'}(sl(3))$ contains the Abelian subalgebra

$$\begin{array}{lcl}
\Delta_{\mathcal{P}'}(K) & = & K \otimes 1 + 1 \otimes K, \\
\Delta_{\mathcal{P}'}(\sigma) & = & \sigma \otimes 1 + 1 \otimes \sigma.
\end{array} \qquad [K, \sigma] = 0, \tag{4.2}$$

Thus, the additional Reshetikhin twist

$$\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)} = e^{\lambda K \otimes \sigma} \tag{4.3}$$

is applicable to the previously obtained Hopf algebra,

$$U_{\mathcal{P}'}(sl(3)) \stackrel{\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)}}{\longrightarrow} U_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(sl(3)). \tag{4.4}$$

The new twisted algebra $U_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(sl(3))$ is defined by the relations:

$$
\begin{array}{rcl}
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(H_{12}) & = & H_{12}\otimes 1+1\otimes H_{12}+(\lambda K+H)\otimes(e^{-\sigma}-1) \\
 & & \quad +\, E_{23}\otimes E_{12}e^{-(\lambda+1)\sigma}, \\
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(H_{13}) & = & (H_{13}-2(\lambda K+H))\otimes 1+2(\lambda K+H)\otimes e^{-\sigma} \\
 & & \quad +1\ \otimes H_{13}+2E_{23}\otimes E_{12}e^{-(\lambda+1)\sigma}, \\
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(E_{12}) & = & E_{12}\otimes e^{\lambda\sigma}+e^{\sigma}\otimes E_{12}, \\
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(E_{13}) & = & E_{13}\otimes e^{\sigma}+1\otimes E_{13}, \\
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(E_{21}) & = & E_{21}\otimes e^{-\lambda\sigma}+1\otimes E_{21} \\
 & & \quad -\, E_{23}\otimes H_{12}e^{-\lambda\sigma}-(\lambda K+H)\otimes E_{23}e^{-\sigma} \\
 & & \quad +\,(\lambda K+H)E_{23}\otimes(e^{-\lambda\sigma}-e^{-(\lambda+1)\sigma}) \\
 & & \quad -\, E_{23}^2\otimes E_{12}e^{-(2\lambda+1)\sigma}-E_{23}\otimes E_{12}E_{23}e^{-(\lambda+1)\sigma}, \\
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(E_{23}) & = & E_{23}\otimes e^{-\lambda\sigma}+1\otimes E_{23}, \\
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(E_{31}) & = & E_{31}\otimes e^{-\sigma}+1\otimes E_{31}+(\lambda K+H)\otimes H_{13}e^{-\sigma} \\
 & & \quad +\, E_{23}\otimes E_{32}e^{-\lambda\sigma} \\
 & & \quad +\,(1-\lambda K-H)(\lambda K+H)\otimes(e^{-\sigma}-e^{-2\sigma}) \\
 & & \quad -\, E_{21}\otimes E_{12}e^{-(\lambda+1)\sigma} \\
 & & \quad +\,(1-\lambda K-H)E_{23}\otimes E_{12}e^{-\lambda\sigma}(e^{-\sigma}-2e^{-2\sigma}) \\
 & & \quad +\, E_{23}\otimes H_{13}E_{12}e^{-(\lambda+1)\sigma}+E_{23}^2\otimes E_{12}^2e^{-2(\lambda+1)\sigma}, \\
\Delta_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(E_{32}) & = & E_{32}\otimes e^{(\lambda-1)\sigma}+1\otimes E_{32}+(\lambda+1)K\otimes E_{12}e^{-\sigma}.
\end{array}
\tag{4.5}
$$

According to the associativity of twisting transformations the same parameterized set of algebras could be obtained directly from $U(sl(3))$ using the composite twist

$$
\mathcal{F}_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}=\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)}\Phi_{\mathcal{E}'}\Phi_j=e^{\lambda K\otimes\sigma}e^{-E_{23}\otimes E_{12}}e^{H\otimes\sigma}. \tag{4.6}
$$

This twisting element can be written in the form

$$
\mathcal{F}_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}=e^{-E_{23}\otimes E_{12}e^{-\lambda\sigma}}e^{(H+\lambda K)\otimes\sigma}. \tag{4.7}
$$

The latter is the extended twist for the Lie algebra

$$
\begin{array}{l}
[H+\lambda K,E_{13}]=E_{13},\quad [H+\lambda K,E_{12}]=\lambda E_{12},\quad [H+\lambda K,E_{23}]=(1-\lambda)E_{23}, \\
[E_{12},E_{23}]=E_{13},\quad [E_{12},E_{13}]=[E_{23},E_{13}]=0.
\end{array}
\tag{4.8}
$$

The relations (4.7) and (4.8) signify that the family $\{U_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(sl(3))\}$ is the complete set of twisted Hopf algebras related to the Frobenius subalgebras $\{\mathbf{L}_{\mathcal{E}(\alpha,\beta)}\in sl(3)\}$ and that $\lambda=\alpha$.

It must be also stressed that the appropriate Reshetikhin twist of the type $\mathcal{F}_{\widetilde{\mathcal{R}}(\lambda)}$ can be constructed for any algebra $U_{\mathcal{E}(\alpha,\beta)}(sl(3))$ — there always exists a Cartan element whose dual is orthogonal to the root ν_E.

Note that any triple of roots $\{\alpha, \beta, \gamma \mid \alpha + \beta = \gamma\}$ of the $sl(3)$ root system can play the role of the triple $\{\nu_{12}, \nu_{23}, \nu_{13}\}$ that was selected in our case to form the carrier subalgebra. The formulas above are irrelevant to this choice, only the interrelations of roots are important. In $sl(3)$ there always exists the equivalence transformation of the root system that identify any such triple with the fixed one.

The obtained set of Hopf algebras corresponds to the parameterized family $r_{\mathcal{E}'(\theta)}$ of r–matrices

$$r_{\mathcal{E}'(\theta)} = E_{23} \wedge E_{12} + \frac{1}{2} E_{13} \wedge H_{13} + \frac{1}{2}\theta E_{13} \wedge (H_{12} - H_{23}), \tag{4.9}$$

where we use the parameter $\theta = \frac{1}{3}(2\lambda - 1)$ measuring the deviation of the extended twist from the canonical rather than from the peripheric one.

Algebras $U_{\mathcal{P}'\widetilde{\mathcal{R}}(\lambda)}(sl(3))$ are the quantizations of the Lie bialgebras $(sl(3), \mathfrak{g}^*_{\mathcal{E}'(\theta)})$. The compositions of $\mathfrak{g}^*_{\mathcal{E}'(\theta)}$ are easily derived with the help of (4.9):

$$\begin{array}{lcl lcl}
[X_{11}, X_{12}] & = & \frac{1}{2}(1+\theta)X_{32}, & [X_{11}, X_{22}] & = & \theta X_{31}, \\
[X_{11}, X_{23}] & = & \frac{1}{2}(1-\theta)X_{21}, & [X_{11}, X_{13}] & = & -\frac{1}{2}(\theta+1)(X_{11} - X_{33}), \\
[X_{11}, X_{33}] & = & -\theta X_{31}, & [X_{12}, X_{13}] & = & \frac{1}{2}(3\theta - 1)X_{12}, \\
[X_{12}, X_{21}] & = & X_{31}, & [X_{12}, X_{23}] & = & -(X_{11} - X_{33}), \\
[X_{12}, X_{22}] & = & (\theta+1)X_{32}, & [X_{12}, X_{33}] & = & -\frac{1}{2}(\theta+1)X_{32}, \\
[X_{13}, X_{21}] & = & \frac{1}{2}(3\theta+1)X_{21}, & [X_{13}, X_{22}] & = & -\theta(X_{11} - X_{33}), \\
[X_{13}, X_{23}] & = & \frac{1}{2}(3\theta+1)X_{23}, & [X_{13}, X_{31}] & = & X_{31}, \\
[X_{13}, X_{32}] & = & \frac{1}{2}(1-3\theta)X_{32}, & [X_{13}, X_{33}] & = & \frac{1}{2}(\theta-1)(X_{11} - X_{33}), \\
[X_{22}, X_{23}] & = & (\theta-1)X_{21}, & [X_{22}, X_{33}] & = & \theta X_{31}, \\
[X_{23}, X_{32}] & = & X_{31}, & [X_{23}, X_{33}] & = & \frac{1}{2}(\theta-1)X_{21}.
\end{array} \tag{4.10}$$

5 Multiparametric Drinfeld–Jimbo and $\mathcal{ET}$ quantizations

The twisting element for the Reshetikhin twist [13] for $U_{\mathcal{DJ}}(sl(3))$,

$$\mathcal{F}_{\mathcal{R}} = e^{\eta H_{23} \wedge H_{12}}, \tag{5.1}$$

converts $U_{\mathcal{DJ}}(sl(3))$ into the twisted algebra $U_{\mathcal{DJR}}(sl(3))$ with the r–matrix of the form

$$r_{\mathcal{DJR}} = r_{\mathcal{DJ}} + r_{\mathcal{R}} = r_{\mathcal{DJ}} + \eta H_{12} \wedge H_{23} = r_{\mathcal{DJ}} + \eta(E_{11} \wedge E_{33} - E_{11} \wedge E_{22} - E_{22} \wedge E_{33}). \tag{5.2}$$

This signifies that the corresponding dual Lie algebra $\mathfrak{g}^*_{\mathcal{DJR}}$ is the first order deformation of $\mathfrak{g}^*_{\mathcal{DJ}}$ by $\mathfrak{g}^*_{\mathcal{R}}$ and η can be viewed as a deformation parameter. The nonzero compositions of $\mathfrak{g}^*_{\mathcal{R}}$ are the following ones :

$$
\begin{array}{rclrcl}
[X_{11}, X_{12}] &=& -X_{12}, & [X_{11}, X_{21}] &=& X_{21}, \\
[X_{22}, X_{12}] &=& -X_{12}, & [X_{22}, X_{21}] &=& X_{21}, \\
[X_{33}, X_{12}] &=& 2X_{12}, & [X_{33}, X_{21}] &=& -2X_{21}, \\
[X_{11}, X_{13}] &=& X_{13}, & [X_{11}, X_{31}] &=& -X_{31}, \\
[X_{22}, X_{13}] &=& -2X_{13}, & [X_{22}, X_{31}] &=& 2X_{31}, \\
[X_{33}, X_{13}] &=& X_{13}, & [X_{33}, X_{31}] &=& -X_{31}, \\
[X_{11}, X_{23}] &=& 2X_{23}, & [X_{11}, X_{32}] &=& -2X_{32}, \\
[X_{22}, X_{23}] &=& -X_{23}, & [X_{22}, X_{32}] &=& X_{32}, \\
[X_{33}, X_{23}] &=& -X_{23}, & [X_{33}, X_{32}] &=& X_{32}.
\end{array}
\tag{5.3}
$$

The compositions $\mu^*_{\mathcal{DJR}}$ of the algebra $\mathfrak{g}^*_{\mathcal{DJR}}$ that was deformed in the first order by $\mu^*_{\mathcal{R}}$ are:

$$
\begin{array}{rclrcl}
[X_{11}, X_{12}] &=& X_{12} - \eta X_{12}, & [X_{11}, X_{21}] &=& X_{21} + \eta X_{21}, \\
[X_{11}, X_{13}] &=& X_{13} + \eta X_{13}, & [X_{11}, X_{31}] &=& X_{31} - \eta X_{31}, \\
[X_{11}, X_{23}] &=& 2\eta X_{23}, & [X_{11}, X_{32}] &=& -2\eta X_{32}, \\
[X_{22}, X_{12}] &=& -X_{12} - \eta X_{12}, & [X_{22}, X_{21}] &=& -X_{21} + \eta X_{21}, \\
[X_{22}, X_{13}] &=& -2\eta X_{13}, & [X_{22}, X_{31}] &=& 2\eta X_{31}, \\
[X_{22}, X_{23}] &=& X_{23} - \eta X_{23}, & [X_{22}, X_{32}] &=& X_{32} + \eta X_{32}, \\
[X_{33}, X_{12}] &=& 2\eta X_{12}, & [X_{33}, X_{21}] &=& -2\eta X_{21}, \\
[X_{33}, X_{13}] &=& -X_{13} + \eta X_{13}, & [X_{33}, X_{31}] &=& -X_{31} - \eta X_{31}, \\
[X_{33}, X_{23}] &=& -X_{23} - \eta X_{23}, & [X_{33}, X_{32}] &=& -X_{32} + \eta X_{23}, \\
[X_{12}, X_{23}] &=& 2X_{13}, & [X_{21}, X_{32}] &=& 2X_{31}.
\end{array}
\tag{5.4}
$$

According to the lemma proved in [11] the necessary and sufficient condition for the existence of a smooth transition connecting two quantized Lie bialgebras $U_q(\mathfrak{g}, \mathfrak{g}^*_1)$ and $U_q(\mathfrak{g}, \mathfrak{g}^*_2)$ is the existence of the first order deformation of μ^*_1 by μ^*_2 (and vice versa). In our case this is the combination of compositions (4.10) and (5.4),

$$
\mu^*(s,t) = s\mu^*_{\mathcal{DJR}}(\eta) + t\mu^*_{\mathcal{E}'}(\theta),
\tag{5.5}
$$

that must be checked. The direct computations show that $\mu^*(s,t)$ is a Lie composition if and only if $\eta = \theta$.

Thus we have proved that for any $U_{\mathcal{E}(\alpha,\beta)}(sl(3))$ there exists one and only one twisted Drinfeld–Jimbo deformation $U_{\mathcal{DJR}(\lambda)}(sl(3))$ that can be connected with the twisted algebra by a smooth path whose points are the deformation quantizations.

Remember that both $\mu^*_{\mathcal{DJR}}(\eta)$ and $\mu^*_{\mathcal{E}'}(\theta)$ are the linear combinations of Lie compositions ($\mu^*_{\mathcal{DJ}}$ and $\mu^*_{\mathcal{R}}$, $\mu^*_{\mathcal{P}'}$ and $\mu^*_{\widetilde{\mathcal{R}}}$). So, we have a four-dimensional space of compositions with two fixed planes of Lie compositions containing correspondingly $\mu^*_{\mathcal{DJR}}(\eta)$ and $\mu^*_{\mathcal{E}'}(\theta)$. From these two planes only the correlated lines (with $\eta = \theta$) belong to the Lie subspaces that intersect both planes.

6 Conclusions

The r–matrix $r_{\mathcal{DJR}}(\eta)$ can be transformed into the mixed r–matrix $r_{\mathcal{DJR}}(\eta) + v\, r_{\mathcal{E}(\eta)}$ with the help of an operator $\exp\{v \,\mathrm{ad}_E\}$ similarly to the ordinary case when $r_{\mathcal{DJ}}$ is transformed into $r_{\mathcal{DJ}} + v\, r^{\mathrm{can}}_{\mathcal{E}}$ [8]. We want to note that the element E may correspond to any root ν of the $sl(3)$ root system. Varying the roots one shall arrive at the r–matrices attributed to different (though equivalent) sets of extended twisted algebras.

The canonically extended twisted algebra $U^{\mathrm{can}}_{\mathcal{E}}(sl(3))$ introduced in [9] is a special case of extended twisted algebras $\{U_{\mathcal{E}'(\alpha,\beta)}(sl(3))\}$. It corresponds to the situation when the functional H^* is parallel to the root ν_E. For the Lie algebras of A_n series this means that $\alpha = \beta = 1/2$. In the Appendix we present the full table of the defining relations for this Hopf algebra.

The peripheric twists helped us to obtain the explicit form of the comultiplication for all the extended twisted Hopf algebras originated from $U(sl(3))$. In the set $\{U_{\mathcal{E}(\alpha,\beta)}(sl(3))\}$ algebras produced by peripheric twists were not distinguished by their relations neither with Drinfeld–Jimbo twists ($\{U_{\mathcal{DJ}}(sl(3))\}$) nor with Reshetikhin twists. We want to note that the situation changes when one studies the specific properties of extensions for peripheric twisted algebras.

The construction presented in this paper can be performed for any two-dimensional sublattice of the root lattice of any simple Lie algebra. For any highest root of the "triple" there exists the Cartan generator whose dual is orthogonal to this root. This means that the corresponding special Reshetikhin twist can always be constructed. The same is true also for the so called special injections of $\mathcal{L} \in \mathfrak{g}$. In this case the "triple" will be realized in the root space submerged in that of the initial simple algebra. Whatever the injection is an ordinary Reshetikhin twist can be applied to the $U_{\mathcal{DJ}}(\mathfrak{g})$ to coordinate the properties of $U_{\mathcal{DJR}}(\mathfrak{g})$ and $U_{\mathcal{ER}}(\mathfrak{g})$. The extended twists for different injections and the role of the peripheric twists will be studied in detail in a forthcoming publication.

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Appendix

In [9] the $\mathcal{E}$-twisted algebra $U_{\mathcal{E}}^{\mathrm{can}}(sl(3))$ was introduced and some of its comultiplications where presented explicitly. In the family $\{U_{\mathcal{E}'(\alpha,\beta)}(sl(3))\}$ it corresponds to the case $\alpha = 1/2$. The involution

$$\begin{array}{rclcrcl} E_{12} & \rightleftharpoons & E_{23}, & & E_{32} & \rightleftharpoons & -E_{21}, \\ E_{23} & \rightleftharpoons & -E_{12}, & & H_{12} & \rightleftharpoons & H_{23}, \\ E_{21} & \rightleftharpoons & E_{32}, & & H_{23} & \rightleftharpoons & H_{12}, \end{array} \tag{6.1}$$

transforms $U_{\mathcal{E}'(1/2,1/2)}(sl(3))$ into $U_{\mathcal{E}(1/2,1/2)}(sl(3))$. The full list of defining coproducts for $U_{\mathcal{E}}^{\mathrm{can}}(sl(3))$ can be thus obtained:

$$\begin{array}{rcl} \Delta_{\mathcal{E}}{}^{\mathrm{can}}(H_{23}) & = & H_{23} \otimes 1 + 1 \otimes H_{23} + \frac{1}{2} H_{13} \otimes (e^{-2\widetilde{\sigma}} - 1) - 2\xi E_{12} \otimes E_{23} e^{-3\widetilde{\sigma}}, \\ \Delta_{\mathcal{E}}{}^{\mathrm{can}}(H_{13}) & = & H_{13} \otimes e^{-2\widetilde{\sigma}} + 1 \otimes H_{13} - 4\xi E_{12} \otimes E_{23} e^{-3\widetilde{\sigma}}, \\ \Delta_{\mathcal{E}}{}^{\mathrm{can}}(E_{23}) & = & E_{23} \otimes e^{\widetilde{\sigma}} + e^{2\widetilde{\sigma}} \otimes E_{23}, \\ \Delta_{\mathcal{E}}{}^{\mathrm{can}}(E_{13}) & = & E_{13} \otimes e^{2\widetilde{\sigma}} + 1 \otimes E_{13}, \\ \Delta_{\mathcal{E}}{}^{\mathrm{can}}(E_{32}) & = & E_{32} \otimes e^{-\widetilde{\sigma}} + 1 \otimes E_{32} + 2\xi E_{12} \otimes H_{23} e^{-\widetilde{\sigma}} + \xi H_{13} \otimes E_{12} e^{-2\widetilde{\sigma}} \\ & & \quad - \xi H_{13} E_{12} \otimes (e^{-\widetilde{\sigma}} - e^{-3\widetilde{\sigma}}) - 4\xi^2 E_{12}^2 \otimes E_{23} e^{-4\widetilde{\sigma}} \\ & & \quad - 4\xi^2 E_{12} \otimes E_{23} E_{12} e^{-3\widetilde{\sigma}}, \\ \Delta_{\mathcal{E}}{}^{\mathrm{can}}(E_{12}) & = & E_{12} \otimes e^{-\widetilde{\sigma}} + 1 \otimes E_{12}, \\ \Delta_{\mathcal{E}}{}^{\mathrm{can}}(E_{31}) & = & E_{31} \otimes e^{-2\widetilde{\sigma}} + 1 \otimes E_{31} + \xi H_{13} \otimes H_{13} e^{-2\widetilde{\sigma}} \\ & & \quad + \xi (1 - \frac{1}{2} H_{13}) H_{13} \otimes (e^{-2\widetilde{\sigma}} - e^{-4\widetilde{\sigma}}) - 2\xi E_{32} \otimes E_{23} e^{-3\widetilde{\sigma}} \\ & & \quad + 2\xi E_{12} \otimes E_{21} e^{-\widetilde{\sigma}} \\ & & \quad - 4\xi^2 (1 - \frac{1}{2} H_{13}) E_{12} \otimes E_{23} e^{-\widetilde{\sigma}} (e^{-2\widetilde{\sigma}} - 2e^{-4\widetilde{\sigma}}) \\ & & \quad - 4\xi^2 E_{12} \otimes H_{13} E_{23} e^{-3\widetilde{\sigma}} + 8\xi^3 E_{12}^2 \otimes E_{23}^2 e^{-6\widetilde{\sigma}}, \\ \Delta_{\mathcal{E}}{}^{\mathrm{can}}(E_{21}) & = & E_{21} \otimes e^{-\widetilde{\sigma}} + 1 \otimes E_{21} + \xi (H_{12} - H_{23}) \otimes E_{23} e^{-2\widetilde{\sigma}}. \end{array} \tag{6.2}$$

Note that the deformation parameter ξ and $\widetilde{\sigma} = \frac{1}{2} \ln(1 + 2\xi E)$ had been introduced here to make the correlations with the previous results more transparent.